CSE 599S Proof Complexity
and its Applications

Lecture 5 October 14, 2020

$Res_{tree}(F)$ = size of shortest tree-like res refutation of F

$\underline{Res(F)}$ = " " resolution refutation of F

$w(F)$ = length of longest clause in F

"width"

$width(F)$ = min length of longest clause in any res proof of F

Last time: $Res_{tree}(F) \geq 2^{width(F) - w(F)}$

$(Res_{tree}(F) \leq S \Rightarrow width(F) \leq \log_2 S + w(F))$

New Thm: $Res(F) \leq S$ (n vars) $\Rightarrow width(F) \leq \underline{2\sqrt{2n \cdot \ln(S)}} + w(F)$ $\Rightarrow 2W + w(F)$

$\rightarrow$ Cor: $Res(F) \geq e^{\frac{(width(F) - w(F))^2}{8n}}$.

Proof Idea: set vars to kill all wide clauses in the proof.

Start with a proof of size S

wide = width $\geq W = \underline{\sqrt{2n \ln(S)}}$

**Claim** If $(1-\frac{W}{2n})^k \cdot S < 1$ then any CNF formula $F$ in $n$ vars with a proof having $\le S$ wide clauses has

$$\text{width}(F) \le W + w(F) + k$$

$(1+x) \le e^x$

If we have this for $k = W$

$$\left(1-\frac{W}{2n}\right)^W \le e^{-\frac{W}{2n}\cdot W} = e^{-\frac{W^2}{2n}} = e^{-\ln(S)} \le 1/S$$

$$\therefore \left(1-\frac{W}{2n}\right)^W \cdot S < 1$$

Proof of Claim by induction:

$k=0$: $S<1$ no wide clauses so $\text{width}(F) \le W$ ✓

Assume true for $k-1$.

and $\left(1-\frac{W}{2n}\right)^k \cdot S < 1$

and $\exists$ proof $P$ of $F$ with $\le S$ wide clauses

Choose literal $z$ appearing in $P$ in most wide clauses.

$\ge WS$ literals in wide clauses

$\therefore z$ appears in $\ge \frac{WS}{2n}$ clauses

$z \leftarrow 1$ kills all wide clauses containing $z$.

$P_{z \leftarrow 1}$ refutes $F_{z \leftarrow 1}$

must have $S' \le S - \frac{wS}{2n}$

$= \left(1 - \frac{w}{2n}\right) \cdot S$

$$\left(1-\frac{w}{2n}\right)^{\cancel{k}\, k-1} S' \le \left(1-\frac{w}{2n}\right)^{k} \cdot S < 1$$ wide clauses

IH $\Rightarrow$ width$(F_{x \leftarrow 1}) \le w^* - 1$

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$z \leftarrow 0$ fewer vars $n-1$

Apply IH. to $F_{x \leftarrow 0}$

$P_{x \leftarrow 0}$ has $\le S$ wide clauses

IH. $\Rightarrow$ width$(F_{x \leftarrow 0}) \le w^*$

... , $\bot$ add back $\bar{x}$ wherever it appear in $F$

(1) $\Rightarrow$ resolution derivation of $\bar{x}$ from $F$ of width $w^*$.

(2) No resolve $\bar{x}$ with clauses of $F$ that contain $x$ to get $F_{x\leftarrow 0}$

width $\leq w(F)$

(3) derive $\bot$ from $F_{x\leftarrow 0}$ in width $w^*$

Total width $\leq w^*$ $\boxtimes$

Just like tree-res case

Cor. If there is a resolution refutation of $F$ of size $S$

Can find it in time $O(\sqrt{n \log(S)}) + w(F)$

If ... tree resolution ... of $F$ : $S$

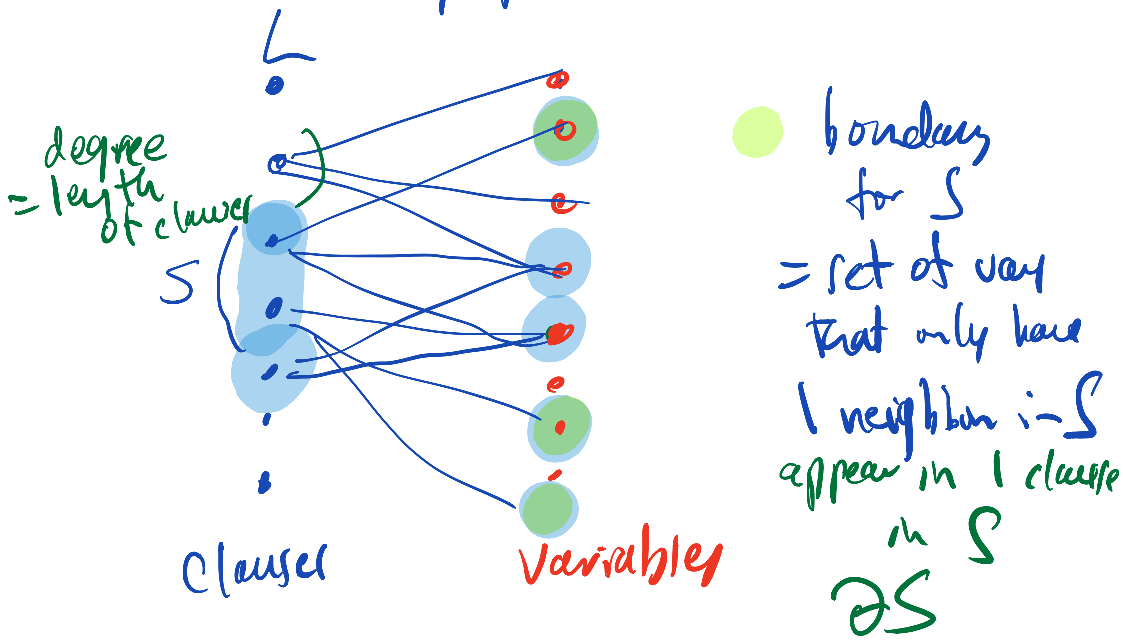
time $n^{\log S + w(F)} = S^{\log n} \cdot n^{w(F)}$

try all possible refutations of size up to width bound.

$2^k \binom{n}{k}$ clauses of width $=k$

# Proving width lower bounds

Bipartite graph for a formula

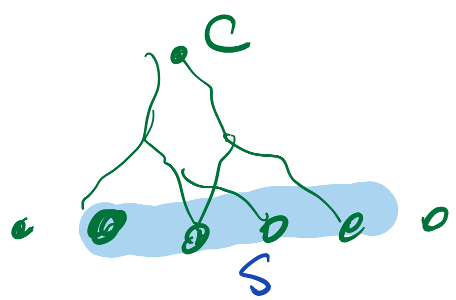



**Defn** Graph is an $(r,c)$ boundary expander iff all $S \subseteq L$ with $|S| \le r$, $|\partial S| \ge c \cdot |S|$

**Lemma** If $S$ is a minimal subset of clauses of $F$ that implies $C$ then $|C| \geq |\partial S|$.

If clauses $S$ are used in a resolution proof to derive $C$ then $|C| \geq |\partial S|$

**Proof**




vars in $\partial S$ can't be cancelled. ∎

**Theorem** If graph of $F$ is an $(r,c)$-boundary expander $\Rightarrow$ $\text{width}(F) \geq cr/2$

**Proof** Given a refutation of $F$ find a target clause $C^*$ that we will show is high complexity $> r$

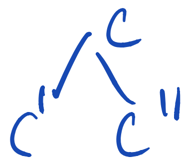

$C$ in $P$
$\text{complexity}(C)$ = # of clauses of $F$ used to derive $C$

**Claim** $\text{complexity}(\perp) > r$

If $\text{complexity}(\perp) \leq r$ by boundary expansion it would have

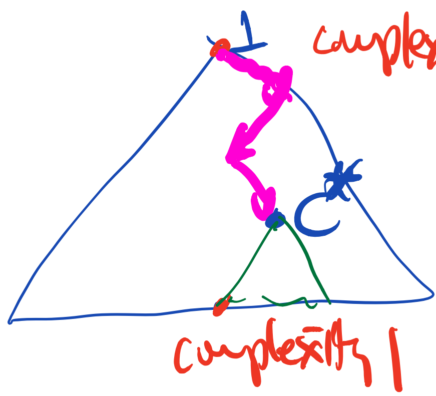
$0 < \text{complexity}(\perp) = |S| \leq r$ clauses used to derive $\perp$

$0 = |\perp| \geq |\partial S| \geq c|S| > 0$ (*)




$$\text{complexity}(C) \le \text{complexity}(C') + \text{complexity}(C'')$$

$$\therefore \text{ complexity}(C') \text{ or } \text{complexity}(C'') \ge \frac{1}{2}\text{complexity}(C)$$




walk back from output choosing larger complexity child until complexity $\le r$

$$\frac{r}{2} < \text{complexity}(C^*) \le r$$

$\frac{r}{2} < |S| \le r$ $\therefore$ Since $F$ is an $(r,c)$ boundary expander

we have $|C^*| \ge c|S| \ge cr/2$ $\blacksquare$